Just for Working Sessions

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We consider some observation operator $H: X\rightarrow Y$ defined on the space X. Our goal is to solve an operator equation of the type \begin{equation} \label{eq org} H \varphi = f \end{equation} with $f \in Y$ given. Here, we think of X as a finite dimensional space. Then, it is isomorphic to $\mathbb{R}^n$, with $n \in \mathbb{N}$. In the same way, the finite dimensional space Y will be isomorphic to $\mathbb{R}^m$ for some $m \in \mathbb{N}$. In general, a linear operator H will consist of sums of multiples of the elements of vectors $\varphi \in X$. If we consider each element φ_j of $$ \varphi = \left(\begin{array}{cc} \varphi_1 \\ \vdots \\ \varphi_n \end{array} \right) $$ for $j=1,\ldotsn$ to belong to a particular point x_j in physical space $\mathbb{R}^d$ of dimension $d \in \{1,2,3\}$, then in general the operator H is not *local* in the sense that its outcome belongs to individual points in space and only depends on the input in these points. In general, the space Y will not be local in the sense that each of its variables belongs to one and only one point in the physical space $\mathbb{R}^d$.

Definition. We call an operator H *local*, if for each variable f_{ξ} for $\xi=1,\ldotsm$ there is at most one point x_{j} in $\mathbb{R}^d$ such that f_{ξ} under the operation of H is influenced by variables φ_j only if they belong to the point x_{j}.

Examples. Consider the matrix \begin{equation} A = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) := \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \end{equation} where φ_{1} and φ_{2} do belong to two different points x_1 and x_2 in physical space $\mathbb{R}^2$. Then A is not a local matrix, since the first component of $A\varphi$ is influenced by both φ_1 and φ_2, i.e. by variables located in two different points x_1 and x_2 in space. The matrix \begin{equation} B = \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) := \left(\begin{array}{cc} 0 & 1 \\ 3 & 0 \end{array} \right) \end{equation} however is local, since the output f_1 is only influenced by φ_2 which is located at x_2 and f_{2} is only influenced by φ_1 located at x_1. The matrix \begin{equation} \label{C example} C = \left(\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right) := \left(\begin{array}{cc} 1 & 0 \\ 3 & 0 \end{array} \right) \end{equation} is local as well, since both output variables f_{1} and f_{2} are influenced by φ_1 only.

Lemma. If we have a local operator for which each measusment is influenced by a different point $x_{j} \in \mathbb{R}^d$, then by reordering of the variables it can be transformed into a diagonal operator. In general, when a point influences two or more output variables, diagonalization by reordering is not possible.

Remark. A reordering operation is equivalent to the application of a permutation matrix P, i.e. a matrix which has exactly one element 1 in each row and column, with all other elements zero.

Proof. We first assume that in the state space $X = \mathbb{R}^n$ each element belong to one and only one point x_{j}, $j=1,\ldots,n$ with $x_{j}\in \mathbb{R}^d$, where all x_{j} are different. Then, each column of the matrix is multiplied with a variable which belongs to a different point $x_j \in \mathbb{R}^d$. The output variable f_{ℓ}, $\ell=1,\ldots,m$ is influenced by the entries in the ℓ-th row of the matrix H. Thus, this is local if and only if at most one of the entries is non-zero. This applies to every row, i.e. in each row there is at most one non-zero element. By the assumption that different measurements are influenced by different points, this means that there can be at most one nonzero entry in each column as well. But that means that the operator H looks like a scaled version of a permutation matrix P, with scaling 0 allowed. Clearly, by reordering we can make this into a diagonal matrix. In general, we take (\ref{C example}) as counter example, and the proof is complete $\Box$

Question. Our question is: can we find transformations $T: X \rightarrow X$ of X and $S: Y \rightarrow Y$ of Y, such that $\tilde{H} := S H T^{-1}$ is local?

An approach using Singular Value Decomposition. By singular value decomposition SVD we have \begin{equation} \label{svd} H = U \Lambda V^{T}, \end{equation} where Λ is a digonal matrix and the matrices U and V consist of orthonormal vectors. When V^T and U are invertible, we can proceed as follows. Now, we define $$ T := V^{T} $$ and $$ S = U^{-1} $$ Then, we have \begin{equation} S H T^{-1} = U^{-1} H (V^{T})^{-1} = \Lambda. \label{work1} \end{equation} We have found the desired transformation by SVD. $\Box$

$$ \tilde{\varphi} = T\varphi$$

$$ H\varphi = HT^{-1}T\varphi $$

$$ = H^{\sim}T\varphi = H^{\sim}\varphi^{\sim} $$

such that $H^{\sim}$ is local